Most Reals Are Fake

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Abstract

This is a rant on the real numbers, uncountable sets, and uncountable infinities. There are at most countably many things that can be defined using finite information. Any uncountable set requires some infinite source of randomness.

How Real are the Reals?

The real numbers have a long history of "shocking" properties [1], of being controversial [2], and of being blamed to be incompatible with physics [3]. Turing [4] established the idea of *computable numbers*, which behave a bit more intuitively. Later, the more general concept of *definable numbers* was developed. In this work, we introduce a simple and coherent perspective on the real numbers and uncountable sets. In particular, we discuss what makes a set uncountable.

We are aware that this is an emotional topic for many mathematicians, so we are trying to go easy on you.

Finite and Infinite Numbers

There are two distinct kinds of real numbers:

1. Finite numbers. These are all numbers definable with *finite* information. For example, $\frac{1}{3}, \pi, e$, and even all non-computable *definable numbers* like Chaitin's constant. The set of all numbers that can be defined using finite means is countable. Any language can express at most countably many numbers, when using finitely many words per number.

2. Infinite numbers. Those are all other real numbers and each of them requires an *infinite* definition. The only way to define any of them is an infinite source of entropy that produces random digits until the end of time and beyond. The uncountable real numbers have to contain uncountably many such numbers having incompressible infinite definitions.

The same is true for Dedekind cuts. There can be at most countably many Dedekind cuts with a finite definition. This also applies to power sets of infinite sets. Any uncountable set has to contain uncountably many elements with an infinite definition.

Kolmogorov Complexity

A set can have at most countably many elements with a finite definition. Every uncountable set has to contain uncountably many elements with infinite definitions. In other words, there are only countably many strings with a *finite* Kolmogorov complexity, but uncountably many strings with an *infinite* Kolmogorov complexity. So there are only countably many definitions of anything. There are at most countably many things that we can think or speak of.

Cantor's Diagonal Argument

Cantor's second diagonal argument shows that for any enumeration of the numbers in the interval $(0, 1)$ he can come up with a new number that is not included in this enumeration. However, computable numbers are not enumerable. That means there is no finite definition for such an enumeration, so it requires an infinite definition. That implies that Cantor's number has to be an infinite number. He implicitly defined a number with an infinite definition by *assuming* an enumeration of the numbers in $(0, 1)$ but considering that every such enumeration requires infinite information shows that Cantor's number has an infinite definition. Cantor defines an infinite number by assuming an enumeration of an unenumerable set.

Any construction of the reals [5] runs into some instance of that same problem and assumes infinite information at some point.

Conclusion

Countability is tightly linked to finite information. The set of all things that can be expressed with finitely many words is countable. Therefore, all uncountable sets have to contain uncountably many elements with an infinite definition. Only infinite things can be uncountable.

If there are no things that represent infinite information then all things in the universe must be countable and therefore it is discrete. Only if there are things representing infinite information the universe is like the reals. Definitely, we can never measure infinite information, so it is impossible to falsify the assumption of infinite things. Thus, modeling physics using uncountable reals implies a strong assumption that we can't verify.

References

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